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Shear correction factors and an energy-consistent beam theory

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Abstract

Presented here is a new derivation of shear correction factors for isotropic beams by matching the exact shear stress resultants and shear strain energy with those of the equivalent first-order shear deformation theory. Moreover, a new method of deriving in-plane and shear warping functions from available elasticity solutions is shown. The derived exact warping functions can be used to check the accuracy of a two-dimensional sectional finite-element analysis of central solutions. The physical meaning of a shear correction factor is shown to be the ratio of the geometric average to the energy average of the transverse shear strain on a cross section. Examples are shown for circular and rectangular cross sections, and the obtained shear correction factors are compared with those of Cowper (1966). The energy-averaged shear representative is also used to derive Timoshenko's beam theory. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

The cross section of a beam may undergo in-plane warping due to extension and bending deformation and out-of-plane warping due to torsion and shear deformation. Moreover, if the beam is anisotropic, in-plane and out-of-plane warping may couple. Then extension can introduce out-of-plane warping and torsion and shear can introduce in-plane warping. Since warpings are small displacements relative to the rigidly translated and rotated cross section, inertia terms due to warpings are relatively small and can be neglected. However, since warpings offer extra degrees of freedom in which the cross section can deform, they influence the structural stiffness and need to be accounted for.

In the literature, transverse normal stresses σ_{22} and σ_{33} and in-plane shear stress σ_{23} are usually assumed to be zero in the constitutive equations to account for in-plane warpings and their influence on the material stiffnesses. The torsional rigidity of a beam with a non-circular cross

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section is usually modified to account for out-of-plane warping due to torsion (see, e.g., Timoshenko and Goodier, 1970). To account for out-of-plane warping due to shear, shear correction factors are used with the first-order shear deformation theory. The deformed cross section is still assumed to be flat and perpendicular to the deformed reference line after the effects of the in-plane warpings and the out-of-plane warping due to torsion are accounted for. On the other hand, the first-order shear deformation theory assumes that the deformed cross section is still flat, but not perpendicular to the deformed reference line. However, the physical meaning of the shear correction factor and the representative shear rotation angle are not well defined in the literature.

Shear effects are significant for thick, sandwich, and built-up beams. Shear effects are also important even for thin laminated composite beams since the ratios of the Young's moduli to the shear moduli are between 20 and 50 in modern composites and between 2.5 and 3.0 in isotropic materials. For plates and shells, there are several shear deformation theories, such as the first-order, third-order (Reddy and Liu, 1985), and layer-wise higher-order theories (Pai et al., 1993; Pai, 1995). All of these shear warping functions are functions of the thickness coordinate only. However, for beams, because shear warping functions are affected by in-plane warpings, especially if the cross section is not rectangular, they are functions of the two coordinates on the cross section. Hence, shear warping functions for two-dimensional structures are not appropriate for beams.

In the first-order shear deformation theory, only linear functions of the two in-plane coordinates are involved in the displacement field and hence exact structural matrices in finite-element analyses can be obtained without using direct numerical integrations, and only C^0 continuity is required for the shear variables if the influence of shear deformation on the axial strain is neglected. Since these advantages are very useful in simplifying the development of a large-scale finite-element code, it is worthwhile to derive shear correction factors that can account for shear effects accurately. However, the following questions arise. How can shear warping functions be derived or calculated? What is the actual physical meaning of the shear representative? Is a shear representative the geometric average or the energy average of the shear strain on a cross section? Is there a way to obtain the shear correction factor without first solving a beam problem with specific boundary and loading conditions?

In the literature, several approaches have been proposed for obtaining the shear correction factor. Most of these approaches are based on matching certain gross responses predicted by the first-order theory with those obtained from the three-dimensional elasticity theory. Gross responses used for matching include the transverse shear strain energy, the propagation velocity of a flexural wave, the natural frequency of the thickness shear vibration mode, and others (Yang et al., 1966; Chow, 1971; Dong and Tso, 1972; Whitney, 1973; Bert, 1983). All these methods require solving the elasticity equilibrium equations with specified boundary and loading conditions, which is difficult for practical use.

In this paper, we present a method of deriving analytical shear warping functions of isotropic beams by using available elasticity solutions for stress distributions. We also present a new method of deriving accurate shear correction factors for isotropic beams by matching the exact shear stress resultants and shear strain energy with those of the equivalent first-order shear deformation theory. This is done without solving the elasticity equilibrium equations with specified boundary and loading conditions. Examples are shown for circular and rectangular cross sections, and the new shear correction factors are compared with those of Cowper (1966).

2. Warping functions

Here we show how to derive the shear warping and coupling functions of prismatic isotropic beams from the elasticity solutions of stress distributions. For an initially straight beam undergoing deformations in three-dimensional space (see Fig. 1), the displacement field can be represented as (Pai and Nayfeh, 1994)

$$\begin{aligned} u_1(x,y,z,t) &= u(x,t)+z\theta_2(x,t)-y\theta_3(x,t) \\ &\quad +\rho_1(x,t)g_{11}(y,z)+\gamma_5(x,t)g_{15}(y,z)+\gamma_6(x,t)g_{16}(y,z) \\ u_2(x,y,z,t) &= v(x,t)-z\theta_1(x,t)+\rho_2(x,t)g_{22}(y,z)+\rho_3(x,t)g_{23}(y,z)+e(x,t)g_{24}(y,z) \\ u_3(x,y,z,t) &= w(x,t)+y\theta_1(x,t)+\rho_2(x,t)g_{32}(y,z)+\rho_3(x,t)g_{33}(y,z)+e(x,t)g_{34}(y,z) \end{aligned} \tag{1}$$

where u_1, u_2 and u_3 are the displacement of an arbitrary point on the observed cross section along the axes, x, y and z, respectively, and t is time. Moreover, u, v and w are the displacements of the area centroid of the observed cross section; θ_1, θ_2 and θ_3 are the rotation angles of the cross section; and ρ_1, ρ_2 and ρ_3 are the curvatures with respect to the axes x, y and z, respectively. e is the extensional strain of the centroidal line. γ_5 and γ_6 are the shear rotation angles at the area centroid with respect to the axes y and $-z$, respectively. g_{11} is the torsional warping function; g_{15} and g_{16} are shear warping functions; g_{22}, g_{23}, g_{32} and g_{33} are bending-induced in-plane warping functions; and g_{24} and g_{34} are extension-induced in-plane warping functions.

Using $\varepsilon_{ii} = \partial u_i/\partial x_i$ and $\varepsilon_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$ ($x_1 \equiv x$, $x_2 \equiv y$, and $x_3 \equiv z$), engineering strains ε_{ij} are obtained as

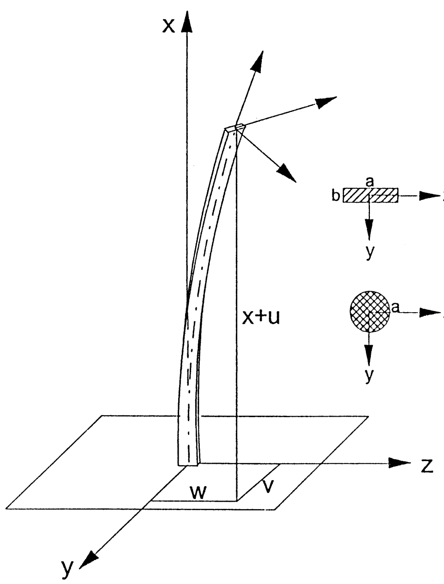

Fig. 1. Coordinate system and displacements for an initially straight beam (xyz is a rectangular frame with the x axis along the beam centroidal line).

$$\varepsilon_{11} = e + z\rho_2 - y\rho_3 + \rho_1' g_{11} + \gamma_5' g_{15} + \gamma_6' g_{16} \tag{2}$$

$$\varepsilon_{22} = \rho_2 g_{22y} + \rho_3 g_{23y} + e g_{24y} \tag{3}$$

$$\varepsilon_{33} = \rho_2 g_{32z} + \rho_3 g_{33z} + e g_{34z} \tag{4}$$

$$\varepsilon_{12} = \rho_1 (g_{11y} - z) + \gamma_5 g_{15y} + \gamma_6 g_{16y} + \rho_2' g_{22} + \rho_3' g_{23} + e' g_{24} \tag{5}$$

$$\varepsilon_{13} = \rho_1 (g_{11z} + y) + \gamma_5 g_{15z} + \gamma_6 g_{16z} + \rho_2' g_{32} + \rho_3' g_{33} + e' g_{34} \tag{6}$$

$$\varepsilon_{23} = \rho_2 (g_{22z} + g_{32y}) + \rho_3 (g_{23z} + g_{33y}) + e(g_{24z} + g_{34y}) \tag{7}$$

where $()' \equiv \partial()/\partial x$, $e = u'$, $\theta_2 = -w'$, $\theta_3 = v'$, $\rho_1 = \theta_1'$, $\rho_2 = \theta_2'$, and $\rho_3 = \theta_3'$

2.1. *In-plane warping functions*

To show the method of obtaining analytical in-plane warping functions, we consider isotropic beams with a cross section symmetric with respect to the axes y and z. We also assume that all loads are applied at the ends and hence $\rho_1' = \gamma_5' = \gamma_6' = e' = 0$. Using the assumption that $\sigma_{22} = \sigma_{33} = \sigma_{23} = 0$ in the constitutive equation of isotropic materials yields $\sigma_{11} = E\varepsilon_{11}$, $\varepsilon_{22} = -v\varepsilon_{11}$, $\varepsilon_{33} = -v\varepsilon_{11}$, and $\varepsilon_{23} = 0$. Here σ_{ij} denote engineering stresses, E is Young's modulus, and v is Poisson's ratio. Using these results and eqns (2)–(4) and (7), we obtain

$$\rho_2 (g_{22y} + vz) + \rho_3 (g_{23y} - vy) + e(g_{24y} + v) = 0 \tag{8}$$

$$\rho_2 (g_{32z} + vz) + \rho_3 (g_{33z} - vy) + e(g_{34z} + v) = 0 \tag{9}$$

$$\rho_2 (g_{22z} + g_{32y}) + \rho_3 (g_{23z} + g_{33y}) + e(g_{24z} + g_{34y}) = 0 \tag{10}$$

Since ρ_2, ρ_3 and e are independent of each other, setting their coefficients in eqns (8)–(10) to zero yields

$$g_{22y} + vz = 0, \quad g_{32z} + vz = 0, \quad g_{22z} + g_{32y} = 0 \tag{11a,b,c}$$

$$g_{23y} - vy = 0, \quad g_{33z} - vy = 0, \quad g_{23z} + g_{33y} = 0 \tag{12a,b,c}$$

$$g_{24y} + v = 0, \quad g_{34z} + v = 0, \quad g_{24z} + g_{34y} = 0 \tag{13a,b,c}$$

Moreover, because the cross section is symmetric with respect to both the y and z axes, we have

$$g_{32}(y,z) = g_{32}(-y,z), \quad g_{23}(y,z) = g_{23}(y,-z), \quad g_{34}(y,z) = g_{34}(-y,z) \tag{14a,b,c}$$

Integrating eqns (11)–(13) and using eqns (14a,b,c,), we obtain the in-plane warping functions as

$$g_{22} = -vyz, \quad g_{23} = \tfrac{1}{2}v(y^2 - z^2), \quad g_{24} = -vy$$

$$g_{32} = \tfrac{1}{2}v(y^2 - z^2), \quad g_{33} = vyz, \quad g_{34} = -vz \tag{15}$$

2.2. *Shear warping functions*

To show the method of deriving shear warping functions, we consider a prismatic isotropic beam with the reference axis x representing the line through the area centroids of the beam. To avoid complications arising from bending-torsion coupling, we assume that the cross section and applied static end loads are symmetric with respect to the x–z plane and hence, the x–z plane is the plane of deflection and

$$v = \gamma_6 = \rho_3 = \theta_1 = \rho_1 = 0 \tag{16a}$$

External loads are assumed to be at the ends only, and hence,

$$\gamma_5' = e' = 0 \tag{16b}$$

Substituting eqns (16a,b) into eqns (2), (5) and (6) yields

$$\varepsilon_{11} = e + z\rho_2, \quad \varepsilon_{12} = \gamma_5 g_{15y} + \rho_2' g_{22}, \quad \varepsilon_{13} = \gamma_5 g_{15z} + \rho_2' g_{32} \tag{17a,b,c}$$

The exact distribution of transverse shear stresses in a uniformly loaded beam is the same as in a tip-loaded cantilever and is given by (Love, 1944; Muskhelishvili, 1963)

$$\sigma_{13} = -\frac{F_3}{2(1+v)I_{22}}\left(\frac{\partial\chi}{\partial z} + \frac{1}{2}vz^2 + \frac{1}{2}(2-v)y^2\right) \tag{18}$$

$$\sigma_{12} = -\frac{F_3}{2(1+v)I_{22}}\left(\frac{\partial\chi}{\partial y} + (2+v)yz\right) \tag{19}$$

where $\chi(y,z)$ is a harmonic function determined by the shape of the cross section and $I_{22} \equiv \iint z^2\,\mathrm{d}y\,\mathrm{d}z$. Moreover, F_3 is the shear stress resultant, which is equal to the end load in the tip-loaded case and varies linearly in a uniformly-loaded case. Since eqns (18) and (19) are exact in the cases of constant and linearly varying F_3, it is expected that eqns (18) and (19) are valid if F_3 does not vary too rapidly along the length of the beam.

2.2.1. *Circular cross sections*

For a circular cross section with a radius a, the function χ is (Love, 1944)

$$\chi = -\tfrac{1}{4}(3+2v)a^2 z + \tfrac{1}{4}(z^3 - 3zy^2) \tag{20}$$

Substituting eqn (20) into eqns (18) and (19) yields

$$\sigma_{13} = \frac{F_3(3+2v)}{8(1+v)I_{22}}\left(a^2 - z^2 - \frac{1-2v}{3+2v}y^2\right) \tag{21}$$

$$\sigma_{12} = -\frac{F_3(1+2v)}{4(1+v)I_{22}}yz \tag{22}$$

Since $\sigma_{13}|_{y=z=0} \equiv G\gamma_5$ (G is the shear modulus), it follows from eqn (21) that

$$G\gamma_5 = \frac{F_3(3+2v)}{8(1+v)I_{22}}a^2 \tag{23}$$

It follows from eqn (17a) that the bending moment $M_2 \equiv \int \sigma_{11} z\,\mathrm{d}y\,\mathrm{d}z = \int E\varepsilon_{11} z\,\mathrm{d}y\,\mathrm{d}z = EI_{22}\rho_2$. Moreover, $M_2 = -\hat{F}_3(L-x)$ (L is the beam length) and $F_3 = \hat{F}_3$ for a cantilever subjected to an end force $\hat{F}_3$, and $M_2 = -q(L-x)^2/2$ and $F_3 = q(L-x)$ for a cantilever subjected to a constant distributed load q. Hence, we obtain that

$$\rho_2' = \frac{M_2'}{EI_{22}} = \frac{F_3}{EI_{22}} \tag{24}$$

Substituting eqn (24) into eqn (17c) and using the relations $\sigma_{13} = G\varepsilon_{13}$ and $E = 2G(1+\nu)$ and eqns (21) and (23) yields

$$\frac{F_3(3+2\nu)}{8(1+\nu)I_{22}}\left(a^2 - z^2 - \frac{1-2\nu}{3+2\nu}y^2\right) = \frac{F_3(3+2\nu)}{8(1+\nu)I_{22}}a^2\left(g_{15z} + \frac{4}{(3+2\nu)a^2}g_{32}\right) \tag{25}$$

It follows from eqns (25) and (15) that

$$g_{15z} = -\frac{y^2}{(3+2\nu)a^2} + 1 - \frac{3z^2}{(3+2\nu)a^2} \tag{26}$$

and, hence

$$g_{15} = -\frac{y^2 z}{(3+2\nu)a^2} + z - \frac{z^3}{(3+2\nu)a^2} + f(y) \tag{27}$$

Similarly, substituting eqn (24) into eqn (17b) and using the constitutive equation $\sigma_{12} = G\varepsilon_{12}$ and eqns (22) and (23) yields

$$-\frac{F_3(1+2\nu)}{4(1+\nu)I_{22}}yz = \frac{F_3(3+2\nu)}{8(1+\nu)I_{22}}a^2\left(g_{15y} + \frac{4}{(3+2\nu)a^2}g_{22}\right) \tag{28}$$

It follows from eqns (28) and (15) that

$$g_{15} = -\frac{y^2 z}{(3+2\nu)a^2} + g(z) \tag{29}$$

It follows from eqns (27) and (29) that $f(y) = 0$ and $g(z) = z - z^3/(3+2\nu)a^2$. Hence, the shear warping function g_{15} is given by

$$g_{15} = -\frac{y^2 z}{(3+2\nu)a^2} + z - \frac{z^3}{(3+2\nu)a^2} \tag{30}$$

It follows from eqns (17b), (17c), (24), (23), and (15) that

$$\varepsilon_{13} = g_{35}\gamma_5, \quad g_{35} = g_{15z} + \frac{\rho_2'}{\gamma_5}g_{32} = 1 - \frac{z^2}{a^2} - \frac{1-2\nu}{(3+2\nu)a^2}y^2 \tag{31}$$

$$\varepsilon_{12} = g_{25}\gamma_5, \quad g_{25} = g_{15y} + \frac{\rho_2'}{\gamma_5}g_{22} = -\frac{2(1+2\nu)}{(3+2\nu)a^2}yz \tag{32}$$

The shear strain functions g_{35} and g_{25} can be directly obtained from eqns (21) and (22) by using the relations $g_{35} = \sigma_{13}/G\gamma_5$ and $g_{25} = \sigma_{12}/G\gamma_5$. If the beam is subjected to shear loads along both y and z directions,

$$\varepsilon_{13} = g_{35}\gamma_5 + g_{36}\gamma_6, \quad \varepsilon_{12} = g_{26}\gamma_6 + g_{25}\gamma_5 \tag{33}$$

where the shear strain functions g_{26} and g_{36} can be obtained by considering end loads along the y axis and are given by

$$g_{26} = 1-\frac{y^2}{a^2}-\frac{1-2v}{(3+2v)a^2}z^2, \quad g_{36} = \frac{2(1+2v)}{(3+2v)a^2}yz \tag{34}$$

2.2.2. *Rectangular cross sections*

For beams having rectangular cross sections (see Fig. 1), the function χ is given by (Love, 1944)

$$\chi = \left(-\frac{1+v}{4}a^2+\frac{v}{12}b^2\right)z+\frac{2+v}{6}(z^3-3zy^2)+\frac{vb^3}{2\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\dfrac{2n\pi z}{b}}{n^3\cosh\dfrac{n\pi a}{b}}\cos\frac{2n\pi y}{b} \tag{35}$$

Substituting eqn (35) into eqns (18) and (19) yields

$$\sigma_{13} = \frac{F_3}{2(1+v)I_{22}}\left[(1+v)\left(\frac{a^2}{4}-z^2\right)-\frac{vb^2}{12}+vy^2-\frac{vb^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cosh\dfrac{2n\pi z}{b}}{n^2\cosh\dfrac{n\pi a}{b}}\cos\frac{2n\pi y}{b}\right] \tag{36}$$

$$\sigma_{12} = \frac{F_3 vb^2}{2(1+v)I_{22}\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh\dfrac{2n\pi z}{b}}{n^2\cosh\dfrac{n\pi a}{b}}\sin\frac{2n\pi y}{b} \tag{37}$$

Since $\sigma_{13}|_{y=z=0} = G\gamma_5$, we obtain from eqn (36) that

$$G\gamma_5 = \frac{F_3}{2(1+v)I_{22}}H_3, \quad H_3 \equiv \frac{1}{4}(1+v)a^2-\frac{1}{12}vb^2-\frac{vb^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\cosh\dfrac{n\pi a}{b}} \tag{38}$$

Using the relations $g_{35} = \sigma_{13}/G\gamma_5$ and $g_{25} = \sigma_{12}/G\gamma_5$, we obtain the shear strain functions g_{35} and g_{25} as

$$g_{35} = \frac{1}{H_3}\left[(1+v)\left(\frac{a^2}{4}-z^2\right)-\frac{vb^2}{12}+vy^2-\frac{vb^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cosh\dfrac{2n\pi z}{b}}{n^2\cosh\dfrac{n\pi a}{b}}\cos\frac{2n\pi y}{b}\right] \tag{39}$$

$$g_{25} = \frac{vb^2}{H_3\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh\dfrac{2n\pi z}{b}}{n^2\cosh\dfrac{n\pi a}{b}}\sin\frac{2n\pi y}{b} \tag{40}$$

Similarly, one can obtain the shear strain function g_{26} and g_{36} as

$$g_{26} = \frac{1}{H_2}\left[(1+v)\left(\frac{b^2}{4}-y^2\right)-\frac{va^2}{12}+vz^2-\frac{va^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cosh\dfrac{2n\pi y}{a}}{n^2\cosh\dfrac{n\pi b}{a}}\cos\frac{2n\pi z}{a}\right] \tag{41}$$

$$g_{36} = -\frac{va^2}{H_2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh\dfrac{2n\pi y}{a}}{n^2\cosh\dfrac{n\pi b}{a}}\sin\frac{2n\pi z}{a} \tag{42}$$

where

$$H_2 \equiv \frac{1}{4}(1+v)b^2-\frac{1}{12}va^2-\frac{va^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\cosh\dfrac{n\pi b}{a}} \tag{43}$$

3. Shear correction factors

For isotropic beams the shear stress–strain relation is

$$\begin{Bmatrix}\sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix}G & 0\\ 0 & G\end{bmatrix}\begin{Bmatrix}g_{35}\gamma_5+g_{36}\gamma_6\\ g_{26}\gamma_6+g_{25}\gamma_5\end{Bmatrix} \tag{44}$$

To derive shear correction factors, we consider the form of eqn (44) and assume that the shear stress resultants F_2 and F_3 of an equivalent first-order shear deformation theory have the form

$$\begin{Bmatrix}F_3\\ F_2\end{Bmatrix} = \begin{bmatrix}k_1GA & k_3GA\\ k_3GA & k_2GA\end{bmatrix}\begin{Bmatrix}\bar{\gamma}_5+c_1\bar{\gamma}_6\\ \bar{\gamma}_6+c_2\bar{\gamma}_5\end{Bmatrix} \tag{45}$$

where A is the cross section area; k_1, k_2 and k_3 are shear correction factors; $\bar{\gamma}_5$ and $\bar{\gamma}_6$ are energy-averaged representatives of γ_5 and γ_6, respectively; k_3 is used to account for any possible coupling of shear energies; c_1 accounts for the shear coupling effect of γ_6 on F_3; and c_2 accounts for the shear coupling effect of γ_5 on F_2. Hence, there are seven unknowns (i.e., k_1, k_2, k_3, $\bar{\gamma}_5$, $\bar{\gamma}_6$, c_1, c_2) to be determined by matching the shear stress resultants F_2 and F_3 and shear strain energy E_n of the exact shear theory with those of the equivalent first-order shear theory.

It follows from eqn (44) that

$$F_3 = \int_A \sigma_{13}\,\mathrm{d}y\,\mathrm{d}z = C_{11}\gamma_5 \tag{46}$$

$$F_2 = \int_A \sigma_{12}\,\mathrm{d}y\,\mathrm{d}z = C_{21}\gamma_6 \tag{47}$$

$$2E_n = \int_A (\sigma_{13}\varepsilon_{13} + \sigma_{12}\varepsilon_{12})\,\mathrm{d}y\,\mathrm{d}z = \gamma_5^2 C_{31} + 2\gamma_6\gamma_5 C_{32} + \gamma_6^2 C_{33} \tag{48}$$

where

$$\int_A g_{36}\,\mathrm{d}y\,\mathrm{d}z = \int_A g_{25}\,\mathrm{d}y\,\mathrm{d}z = 0 \tag{49a}$$

$$C_{11} = G\int_A g_{35}\,\mathrm{d}y\,\mathrm{d}z, \quad C_{21} = G\int_A g_{26}\,\mathrm{d}y\,\mathrm{d}z, \quad C_{31} = G\int_A (g_{25}^2 + g_{35}^2)\,\mathrm{d}y\,\mathrm{d}z$$

$$C_{32} = G\int_A (g_{25}g_{26} + g_{35}g_{36})\,\mathrm{d}y\,\mathrm{d}z, \quad C_{33} = G\int_A (g_{26}^2 + g_{36}^2)\,\mathrm{d}y\,\mathrm{d}z \tag{49b}$$

Equation (49a) is due to the fact that $\int_A \sigma \mathrm{d}A = 0$ when the external load F_2 and hence, γ_6 are zero. Similarly, $\int_A \sigma_{13}\,\mathrm{d}A = 0$ when F_3 and γ_5 are zero.

It follows from eqn (45) that

$$F_3 = (k_1 + c_2k_3)GA\bar{\gamma}_5 + (k_3 + c_1k_1)GA\bar{\gamma}_6 \tag{50}$$

$$F_2 = (k_2 + c_1k_3)GA\bar{\gamma}_6 + (k_3 + c_2k_2)GA\bar{\gamma}_5 \tag{51}$$

$$\begin{aligned} 2E_n &= F_3(\bar{\gamma}_5 + c_1\bar{\gamma}_6) + F_2(\bar{\gamma}_6 + c_2\bar{\gamma}_5) \\ &= \bar{\gamma}_5^2 GA(k_1 + 2k_3c_2 + k_2c_2^2) + 2\bar{\gamma}_6\bar{\gamma}_5 GA[k_1c_1 + k_3(1 + c_1c_2) + k_2c_2] \\ &\quad + \bar{\gamma}_6^2 GA(k_2 + 2k_3c_1 + k_1c_1^2) \end{aligned} \tag{52}$$

Setting the term which contains $\gamma_5(\gamma_6)$ in eqn (46) equal to the term which contains $\bar{\gamma}_5(\bar{\gamma}_6)$ in eqn (50) yields

$$(k_1 + c_2k_3)GA\bar{\gamma}_5 = C_{11}\gamma_5 \tag{53}$$

$$(k_3 + c_1k_1)GA\bar{\gamma}_6 = 0 \tag{54}$$

Similarly, it follows from eqns (47) and (51) that

$$(k_2 + c_1k_3)GA\bar{\gamma}_6 = C_{21}\gamma_6 \tag{55}$$

$$(k_3 + c_2k_2)GA\bar{\gamma}_5 = 0 \tag{56}$$

Also, we obtain from eqns (48) and (52) that

$$\bar{\gamma}_5^2 GA(k_1 + 2k_3c_2 + k_2c_2^2) = C_{31}\gamma_5^2 \tag{57}$$

$$\bar{\gamma}_6\bar{\gamma}_5 GA[k_1c_1 + k_3(1 + c_1c_2) + k_2c_2] = C_{32}\gamma_6\gamma_5 \tag{58}$$

$$\bar{\gamma}_6^2 GA(k_2 + 2k_3c_1 + k_1c_1^2) = C_{33}\gamma_6^2 \tag{59}$$

Substituting eqns (53) and (56) into eqn (58) yields

$$\bar{\gamma}_6 c_1 C_{11} = C_{32}\gamma_6 \tag{60}$$

Substituting eqns (54) and (55) into eqn (59) yields

$$\bar{\gamma}_6 C_{21} = C_{33}\gamma_6 \tag{61}$$

Substituting eqns (53) and (56) into eqn (57) yields

$$\bar{\gamma}_5 C_{11} = C_{31}\gamma_5 \tag{62}$$

Substituting eqns (54) and (55) into eqn (58) yields

$$\bar{\gamma}_5 c_2 C_{21} = C_{32}\gamma_5 \tag{63}$$

It follows from eqns (60)–(63) that

$$c_1 = \frac{C_{21}C_{32}}{C_{11}C_{33}} \tag{64}$$

$$c_2 = \frac{C_{32}C_{11}}{C_{21}C_{31}} \tag{65}$$

Moreover, it follows from eqns (62) and (61) that

$$\frac{\gamma_5}{\bar{\gamma}_5} = \frac{C_{11}}{C_{31}} \tag{66}$$

$$\frac{\gamma_6}{\bar{\gamma}_6} = \frac{C_{21}}{C_{33}} \tag{67}$$

Using eqns (53), (54), (66), and (67), we obtain the shear correction factors as

$$k_1 = \frac{C_{11}^2}{GAC_{31}(1-c_1c_2)} \tag{68}$$

$$k_3 = \frac{-c_1C_{11}^2}{GAC_{31}(1-c_1c_2)} \tag{69}$$

We also obtain from eqns (55), (56), (66), and (67) that

$$k_2 = \frac{C_{21}^2}{GAC_{33}(1-c_1c_2)} \tag{70}$$

$$k_3 = \frac{-c_2C_{21}^2}{GAC_{33}(1-c_1c_2)} \tag{71}$$

It can be proved that the k_3 in eqn (69) is equal to that in eqn (71) by using eqns (64) and (65). It can be seen from eqns (51) and (56) that $F_2 = 0$ if $\bar{\gamma}_6 = 0$. However, eqn (45) shows that, when $\bar{\gamma}_6 = 0$, F_2 can be zero only if k_3 is nontrivial when both k_2 and c_2 are nontrivial.

Substituting eqns (54) and (56) into eqn (45) yields

$$\begin{Bmatrix} F_3 \\ F_2 \end{Bmatrix} = \begin{bmatrix} \bar{k}_1 GA & 0 \\ 0 & \bar{k}_2 GA \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_5 \\ \bar{\gamma}_6 \end{Bmatrix} \tag{72}$$

where

$$\bar{k}_1 \equiv k_1 + c_2 k_3, \quad \bar{k}_2 \equiv k_2 + c_1 k_3 \tag{73}$$

Substituting eqns (68)–(71) into eqn (73) yields

$$\bar{k}_1 = \frac{C_{11}^2}{GAC_{31}}, \quad \bar{k}_2 = \frac{C_{21}^2}{GAC_{33}} \tag{74}$$

The geometric averages of the shear angles are defined as $\tilde{\gamma}_5 \equiv \int_A \varepsilon_{13}\, \mathrm{d}A/A$ and $\tilde{\gamma}_6 \equiv \int_A \varepsilon_{12}\, \mathrm{d}A/A$. It follows from eqns (33), (49a,b), (66), (67), and (74) that

$$\tilde{\gamma}_5 = \frac{C_{11}\gamma_5}{GA} = \bar{k}_1\bar{\gamma}_5, \quad \tilde{\gamma}_6 = \frac{C_{21}\gamma_6}{GA} = \bar{k}_2\bar{\gamma}_6 \tag{75a}$$

Hence, eqn (72) can be rewritten as

$$\begin{Bmatrix} F_3 \\ F_2 \end{Bmatrix} = \begin{bmatrix} GA & 0 \\ 0 & GA \end{bmatrix} \begin{Bmatrix} \tilde{\gamma}_5 \\ \tilde{\gamma}_6 \end{Bmatrix} \tag{75b}$$

Moreover, if $F_2 = 0$ and hence $\bar{\gamma}_6 = 0$, it follows from eqn (52) that

$$2E_n = F_3\bar{\gamma}_5 = GA\tilde{\gamma}_5\bar{\gamma}_5 \neq GA\tilde{\gamma}_5\tilde{\gamma}_5 \neq GA\bar{\gamma}_5\bar{\gamma}_5 \tag{76}$$

Equation (76) shows that $\bar{\gamma}_5$ represents the energy average of ε_{13}. Moreover, eqn (75a) shows that the shear correction factor $\bar{k}_1$ represents the ratio of the geometric average to the energy average of ε_{13}.

The influence of shear warpings on the axial strain ε_{11} is not included in the matching of strain energies. However, if F_2 and F_3 are constant, then $\gamma_5' = \gamma_6' = 0$ and the shear strain energy is decoupled from the axial strain energy. Also, we note that kinetic energy is not considered in the matching. Since kinetic energy is a function of u, v, and w as well as γ_5 and γ_6, the system responses would need to be obtained before the kinetic energies can be matched. This is generally not practical and the results are problem dependent. However, since the kinetic energy due to shear warping is relatively small, using the shear correcting factors obtained by matching only the shear strain energy should not significantly reduce accuracy. The warping-restraint effect can affect the shear warping functions at the ends of a beam if the load distributions on the ends are not the same as those of St Venant's solutions (Iesan, 1987). However, the warping restraint effect is not significant for isotropic beams and is neglected here.

Because shear stress resultants and energy are matched, the corresponding first-order shear theory is energy-consistent. To use this energy-consistent first-order shear theory in solving structural problems, one needs to defined coupled energy-averaged shear rotation angles $\hat{\gamma}_5$ and $\hat{\gamma}_6$ [see eqn (45)]

$$\hat{\gamma}_5 \equiv \bar{\gamma}_5 + c_1\bar{\gamma}_6, \quad \hat{\gamma}_6 \equiv \bar{\gamma}_6 + c_2\bar{\gamma}_5 \tag{77a}$$

Then, the equivalent displacement field for flexural problems is

$$u_1 = -w'z + \hat{\gamma}_5 z - v'y + \hat{\gamma}_6 y, \quad u_2 = v, \quad u_3 = w \tag{77b}$$

Using eqns (77a) and (77b) to derive the first-order shear-deformable beam theory and then solving the governing equations with specified boundary and loading conditions, one can obtain the solutions of v, w, $\bar{\gamma}_5$ and $\bar{\gamma}_6$. After the values of $\bar{\gamma}_5$ and $\bar{\gamma}_6$ are obtained, one can use eqns (66) and

(67) to obtain γ_5 and γ_6, and then eqns (33) and (44) to obtain the transverse shear strains and stresses. After the system responses are obtained by using the equivalent first-order shear deformation theory, if greater accuracy is required, post-processing techniques (Whitney, 1987; Noor and Burton, 1989) can improve the solution by solving the three-dimensional elasticity equations.

4. Energy-consistent formulation

4.1. *Planar flexural vibrations*

We assume that the cross section and applied dynamic loads are symmetric with respect to the x–y plane and that the axis z is a principal axis. Hence, because of the loading condition, $F_2 = \gamma_6 = \bar{\gamma}_6 = 0$. To derive the equations of motion, we consider eqns (77a,b) and rewrite the displacement field as

$$u_1 = -zw' + z\bar{\gamma}_5, \quad u_3 = w, \quad u_2 = 0 \tag{78}$$

The strain–displacement relations are

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x} = -zw'' + z\bar{\gamma}_5', \quad \varepsilon_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \bar{\gamma}_5, \quad \varepsilon_{12} = 0 \tag{79}$$

To derive the equations of motion, we use the extended Hamilton principle, which states

$$0 = \int_0^t (\delta T - \delta V + \delta W_{nc} + \delta W_b)\,\mathrm{d}t \tag{80}$$

where T is the kinetic energy, V is the elastic energy, δW_{nc} is the variation of nonconservative energy due to external loads and damping, and δW_b is the variation of work due to forced applied at the boundary or due to motion of the boundary. Since δW_b is problem dependent, it will not be considered in the derivation. Using the assumptions $\sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, we obtain that

$$\delta W_{nc} = \int_0^L q_3 \delta w\,\mathrm{d}x \tag{81a}$$

$$\delta V = \int_0^L \int_A (\sigma_{11}\delta\varepsilon_{11} + \sigma_{12}\delta\varepsilon_{12} + \sigma_{13}\delta\varepsilon_{13})\,\mathrm{d}A\,\mathrm{d}x \tag{81b}$$

$$\delta T = -\int_0^L \int_A \rho\ddot{\mathbf{D}}\cdot\delta\mathbf{D}\,\mathrm{d}A\,\mathrm{d}x \tag{81c}$$

where q_3 is the external distributed load, ρ is the mass density, and $\mathbf{D}$ is the displacement vector given by

$$\mathbf{D} = u_1\mathbf{i}_x + u_2\mathbf{i}_y + u_3\mathbf{i}_z = (-zw' + z\bar{\gamma}_5)\mathbf{i}_x + w\mathbf{i}_z \tag{82}$$

It follows from eqns (81c) and (82) that

$$\delta T = -\int_0^L \{[m\ddot{w}+j_2(\ddot{\bar{\gamma}}'_5-\ddot{w}'')]\delta w+j_2(\ddot{\bar{\gamma}}_5-\ddot{w}')\delta\bar{\gamma}_5\}\,\mathrm{d}x+j_2(\ddot{\bar{\gamma}}_5-\ddot{w}')\delta w|_0^L \tag{83}$$

where

$$m \equiv \int_A \rho\,\mathrm{d}A, \quad j_2 \equiv \int_A \rho z^2\,\mathrm{d}A \tag{84}$$

Substituting eqn (79) into eqn (81b), we obtain

$$\delta V = \int_0^L [-M''_2\delta w+(F_3-M'_2)\delta\bar{\gamma}_5]\,\mathrm{d}x+[-M_2\delta w'+M'_2\delta w+M_2\delta\bar{\gamma}_5]_0^L \tag{85}$$

where

$$M_2 \equiv \int_A \sigma_{11}z\,\mathrm{d}A = EI_{22}(\bar{\gamma}'_5-w'') \tag{86}$$

Substituting eqns (81a), (83), (85), (86), and (72) into eqn (80) and setting the coefficients of δw and $\delta\bar{\gamma}_5$ to zero, we obtain the following equations of motion:

$$EI_{22}(\bar{\gamma}'''_5-w^{iv})+q_3 = m\ddot{w}+j_2(\ddot{\bar{\gamma}}'_5-\ddot{w}'') \tag{87a}$$

$$EI_{22}(\bar{\gamma}''_5-w''')-\bar{k}_1GA\bar{\gamma}_5 = j_2(\ddot{\bar{\gamma}}_5-\ddot{w}') \tag{87b}$$

The boundary conditions are to specify

$$w \quad \text{or } -M'_2+j_2(\ddot{\bar{\gamma}}_5-\ddot{w}'); \quad w' \quad \text{or } M_2; \quad \bar{\gamma}_5 \quad \text{or } M_2 \tag{88}$$

at $x = 0$, L. In the literature, most authors use the total rotation angle ψ_2 of the observed cross section in the formulation of Timoshenko's beam theory, which is defined as

$$\psi_2 = \bar{\gamma}_5-w' \tag{89}$$

Substituting eqn (87b) into eqn (87a) and using eqn (89) in eqns (87a,b), we obtain that

$$\bar{k}_1GA(w''+\psi'_2)+q_3 = m\ddot{w} \tag{90a}$$

$$EI_{22}\psi''_2-\bar{k}_1GA(w'+\psi_2) = j_2\ddot{\psi}_2 \tag{90b}$$

We note that eqns (90a,b) are the same as those of Timoshenko's beam theory except that ψ_2 represents the energy-averaged rotation angle and the shear correction factor $\bar{k}_1$ accounts for both the shear stress resultant and energy. Because either eqns (87a) and (87b) or eqns (90a) and (90b) are coupled equations, they need to be solved simultaneously. Moreover, although different dependent variables are used (i.e., w vs $\bar{\gamma}_5$ and w vs ψ_2), eqns (87a,b) and eqns (90a,b) describe the same dynamic system, and their solutions are equivalent. Hence, these two formulations are equivalent.

4.2. *Three-dimensional flexural vibrations*

We consider a prismatic homogeneous isotropic beam with the reference axis x being the centroidal line and the axis y and z being the principal axes. The displacement field is shown in eqn (77b). The strain–displacement relations are obtained as

$$\varepsilon_{11} = -zw'' + z\hat{\gamma}'_5 - yv'' + y\hat{\gamma}'_6, \quad \varepsilon_{12} = \hat{\gamma}_6, \quad \varepsilon_{13} = \hat{\gamma}_5 \tag{91}$$

The virtual work due to external loads is

$$\delta W_{nc} = \int_0^L (q_2 \delta v + q_3 \delta w)\,\mathrm{d}x \tag{92}$$

where q_2 is the external distributed load along the y-direction, and q_3 is the external distributed load along the z-direction, and the displacement vector is given by

$$\mathbf{D} = u_1 \mathbf{i}_x + u_2 \mathbf{i}_y + u_3 \mathbf{i}_z = (-w'z + \hat{\gamma}_5 z - v'y + \hat{\gamma}_6 y)\mathbf{i}_x + v\mathbf{i}_y + w\mathbf{i}_z \tag{93}$$

It follows from eqns (81c) and (93) that

$$\begin{aligned}\delta T = -\int_0^L & \{[m\ddot{w} + j_2(\ddot{\hat{\gamma}}'_5 - \ddot{w}'')]\delta w + [j_2(\ddot{\hat{\gamma}}_5 - \ddot{w}') + c_2 j_3(\ddot{\hat{\gamma}}_6 - \ddot{v}')]\delta\bar{\gamma}_5 \\ & + [m\ddot{v} + j_3(\ddot{\hat{\gamma}}'_6 - \ddot{v}'')]\delta v + [j_3(\ddot{\hat{\gamma}}_6 - \ddot{v}') + c_1 j_2(\ddot{\hat{\gamma}}_5 - \ddot{w}')]\delta\bar{\gamma}_6\}\,\mathrm{d}x \\ & + [j_2(\ddot{\hat{\gamma}}_5 - \ddot{w}')\delta w + j_3(\ddot{\hat{\gamma}}_6 - \ddot{v}')\delta v]_0^L\end{aligned} \tag{94}$$

where

$$m \equiv \int_A \rho\,\mathrm{d}A, \quad j_2 \equiv \int_A \rho z^2\,\mathrm{d}A, \quad j_3 \equiv \int_A \rho y^2\,\mathrm{d}A, \quad \int_A \rho yz\,\mathrm{d}A = 0 \tag{95}$$

Substituting eqn (91) into eqn (81b), we obtain

$$\begin{aligned}\delta V = \int_0^L & \{-M''_2 \delta w + M''_3 \delta v + [F_3 - M'_2 + c_2(F_2 + M'_3)]\delta\bar{\gamma}_5 + [F_2 + M'_3 + c_1(F_3 - M'_2)]\delta\bar{\gamma}_6\}\,\mathrm{d}x \\ & + [-M_2 \delta w' + M'_2 \delta w + M_3 \delta v' - M'_3 \delta v + (M_2 - c_2 M_3)\delta\bar{\gamma}_5 - (M_3 - c_1 M_2)\delta\bar{\gamma}_6]_0^L\end{aligned} \tag{96}$$

where

$$\begin{aligned}& M_2 \equiv \int_A \sigma_{11} z\,\mathrm{d}A = EI_{22}(\hat{\gamma}'_5 - w''), \quad M \equiv -\int_A \sigma_{11} y\,\mathrm{d}A = EI_{33}(v'' - \hat{\gamma}'_6) \\ & I_{22} = \int_A z^2\,\mathrm{d}A, \quad I_{33} = \int_A y^2\,\mathrm{d}A\end{aligned} \tag{97}$$

Substituting eqns (92), (94), and (96) into eqn (80), using eqn (97), and setting the coefficients of δw, δv, $\delta\bar{\gamma}_5$, and $\delta\bar{\gamma}_6$ to zero, we obtain the following equations of motion:

$$EI_{22}(\hat{\gamma}'''_5 - w^{iv}) + q_3 = m\ddot{w} + j_2(\ddot{\hat{\gamma}}'_5 - \ddot{w}'') \tag{98a}$$

$$EI_{33}(\hat{\gamma}_6''' - v^{iv}) + q_2 = m\ddot{v} + j_3(\ddot{\hat{\gamma}}_6' - \ddot{v}'') \tag{98b}$$

$$EI_{22}(\hat{\gamma}_5'' - w''') - \bar{k}_1 GA\bar{\gamma}_5 + c_2 EI_{33}(\hat{\gamma}_6'' - v''') - c_2\bar{k}_2 GA\bar{\gamma}_6 = j_2(\ddot{\hat{\gamma}}_5 - \ddot{w}') + c_2 j_3(\ddot{\hat{\gamma}}_6 - \ddot{v}') \tag{98c}$$

$$EI_{33}(\hat{\gamma}_6'' - v''') - \bar{k}_2 GA\bar{\gamma}_6 + c_1 EI_{22}(\hat{\gamma}_5'' - w''') - c_1\bar{k}_1 GA\bar{\gamma}_5 = j_3(\ddot{\hat{\gamma}}_6 - \ddot{v}') + c_1 j_2(\ddot{\hat{\gamma}}_5 - \ddot{w}') \tag{98d}$$

The boundary conditions are to specify

$$\begin{aligned} &w \quad \text{or } -M_2' + j_2(\ddot{\hat{\gamma}}_5 - \ddot{w}') \\ &w' \quad \text{or } M_2 \\ &v \quad \text{or } M_3' + j_3(\ddot{\hat{\gamma}}_6 - \ddot{v}') \\ &v' \quad \text{or } M_3 \\ &\bar{\gamma}_5 \quad \text{or } M_2 - c_2 M_3 \\ &\bar{\gamma}_6 \quad \text{or } M_3 - c_1 M_2 \end{aligned} \tag{99}$$

at $x = 0$, L. Subtracting $c_2 \times$ eqn (98d) from eqn (98c) and subtracting $c_1 \times$ eqn (98c) from eqn (98d) yields

$$EI_{22}(\hat{\gamma}_5'' - w''') - \bar{k}_1 GA\bar{\gamma}_5 = j_2(\ddot{\hat{\gamma}}_5 - \ddot{w}') \tag{100a}$$

$$EI_{33}(\hat{\gamma}_6'' - v''') - \bar{k}_2 GA\bar{\gamma}_6 = j_3(\ddot{\hat{\gamma}}_6 - \ddot{v}') \tag{100b}$$

We note that, when $\bar{\gamma}_6 = 0$, eqn (98a) and eqn (100a) reduce to eqn (87a) and eqn (87b), respectively. However, when both $\bar{\gamma}_5$ and $\bar{\gamma}_6$ are nontrivial, c_1 and c_2 couple the equations governing motion in the two planes.

5. Numerical results and discussion

5.1. *Circular cross section*

Substituting eqns (31), (32) and (34) into eqn (49b), we obtain

$$C_{11} = C_{21} = GA\frac{2(1+v)}{3+2v}, \quad C_{31} = C_{33} = GA\frac{28+56v+32v^2}{6(3+2v)^2}, \quad C_{32} = 0 \tag{101}$$

where $A = a^2\pi$. Substituting eqn (101) into eqns (64)–(67), (73), and (74) yields

$$c_1 = c_2 = k_3 = 0 \tag{102a}$$

$$\frac{\gamma_5}{\bar{\gamma}_5} = \frac{\gamma_6}{\bar{\gamma}_6} = \frac{3(1+v)(3+2v)}{7+14v+8v^2} \tag{102b}$$

$$\bar{k}_1 = \bar{k}_2 = k_1 = k_2 = \frac{6(1+v)^2}{7+14v+8v^2} \tag{102c}$$

The shear correction factor k_1 is the same as that of Cowper (1966) $\{k_{\text{Cowper}} = 6(1+v)/(7+6v)\}$ only if $v = 0$. If $v = 0.3$, k_{Cowper} is higher than k_1 by 4.2%. If $v = 0.5$, k_{Cowper} is higher than k_1 by 6.7%.

5.2. *Rectangular cross section*

Substituting eqns (39) and (40) into eqn (49b) we obtain

$$C_{11} = \frac{GA}{H_3}\frac{a^2(1+v)}{6}, \quad C_{32} = 0, \quad C_{31} = \frac{GA}{H_3^2}\left(\frac{a^4(1+v)^2}{30} + \frac{v^2 b^4}{180} - \frac{v^2 b^5}{2\pi^5 a}\sum_{n=1}^{\infty}\frac{\tanh n\pi a/b}{n^5}\right) \tag{103}$$

where $A = ab$. Substituting eqn (103) into eqns (64)–(67) and (74) yields

$$c_1 = c_2 = k_3 = 0, \quad \bar{k}_1 = k_1 = \frac{a^4(1+v)^2}{36\left(\dfrac{a^4(1+v)^2}{30} + \dfrac{v^2 b^4}{180} - \dfrac{v^2 b^5}{2\pi^5 a}\displaystyle\sum_{n=1}^{\infty}\frac{\tanh n\pi a/b}{n^5}\right)} \tag{104}$$

It is well known in the analysis of isotropic plates that the shear strain functions are given by (Shames and Dym, 1985)

$$g_{35} = 1 - \frac{4z^2}{a^2}, \quad g_{25} = 0 \tag{105}$$

This is the so-called third-order shear deformation theory. Using eqn (105) we obtain

$$C_{11} = \frac{2GA}{3}, \quad C_{31} = \frac{8GA}{15}, \quad \bar{k}_1 = k_1 = \frac{5}{6} \tag{106}$$

If $v = 0$ and/or $b/a \simeq 0$, it follows from eqn (104) that $k_1 = 5/6$. On the other hand, if $v = 0$ and/or $b/a \simeq 0$, it follows from eqns (39) and (40) that g_{35} and g_{25} are the same as those in eqn (105). In other words, neglecting Poisson's effect (i.e., $v = 0$) and/or assuming $b/a \simeq 0$ validates the third-order shear theory. The shear correction factor k_1 in eqn (104) is the same as that of Cowper (1966) $\{k_{\text{Cowper}} = 10(1+v)/(12+11v)\}$ only if $v = 0$. However, k_{Cowper} is independent of a/b, but eqn (104) shows that k_1 is a function of a/b and that k_1 decreases when a/b decreases. For materials with $v = 0.3$, Table 1 shows the comparison. The reason for this is that, when a/b decreases and the beam is subjected to F_3 only, σ_{12} increases [see eqn (37)] due to Poisson's effect and the assumption that $\sigma_{22} = 0$. Hence, the energy averaged shear rotation angle $\bar{\gamma}_5$ and $\bar{\gamma}_5/\bar{\gamma}_5$ $(=1/\bar{k}_1)$ increase. In fact, the shear correction factor of Cowper (1966) for elliptical cross sections also shows this phenomenon. Note that the shear correction factor of a rectangular cross section should be similar to that of an elliptical one when the aspect ratio a/b is very small. However, we point out here that, when a/b is small, the assumption $\sigma_{22} = 0$ is not valid and hence, the shear correction factor [eqn (104)] and the shear stress functions [i.e., eqns (36) and (37)] are not appropriate because they are derived using the assumption $\sigma_{22} = 0$. Hence, if a/b is very small, it is better to obtain k_1 by assuming $\sigma_{12} = g_{25} = 0$. Table 1 shows that $k_{1(\sigma_{12}=g_{25}=0)}$ is larger than that from eqn

Table 1
Shear correction factors for beams having rectangular cross sections and $\nu = 0.3$

a/b	k_1	$k_{1(\sigma_{12}=g_{25}=0)}$	k_{Cowper}
10	5/6	5/6	0.84967
4	0.83331	0.83331	0.84967
2	0.83294	0.83298	0.84967
1	0.82822	0.82930	0.84967
0.5	0.78444	0.80676	0.84967
0.25	0.58370	0.74767	0.84967
0.1	0.17913	0.60987	0.84967

(104) and is less than k_{Cowper}. However, to obtain accurate results, one should treat beams with very small a/b as plates.

5.3. *Discussion*

The shear correction factor defined in this paper is more rigorous than others in the literature because both shear stress resultants and shear strain energy are conserved. The physical meaning of shear correction factor is shown to be the ratio of the geometric average to the energy average of shear strain [eqn (75a)], not the ratio of the geometric average of the shear strain to the shear rotation angle at the centroid as explained in some of the literature. Moreover, if the geometric average is used as the shear representative and only shear stress resultants are matched, the shear correction factor should be one, as shown in eqn (75b). The present shear correction factor is different from that of Cowper (1966) because the geometric average of the shear strain is used by Cowper as the shear representative and eqn (81b) is not satisfied. In other words, the formulation of Cowper (1966) is not energy-consistent.

A combination of the present method of deriving shear warping functions from elasticity solutions and the new derivation of shear correction factor can be used to obtain energy-consistent shear correction factors. Since the range of validity of the first-order shear theory is strongly dependent on the shear correction factors used, the present shear correction factor can be used to extend the validity of the first-order shear theory in analyzing thick beams. These shear correction factors are useful in finite-element formulations to simplify the formulation but still accurately account for shear effects.

For asymmetric cross sections or symmetric anisotropic cross sections, the in-plane warping functions in eqn (15) are not valid. To obtain the warping functions and shear strain functions g_{25}, g_{26}, g_{35} and g_{36} of such complex cross sections, one needs to perform two-dimensional sectional finite-element analyses (see, e.g., Giavotto et al., 1983) to obtain the shear stress distributions σ_{13} and σ_{12} and then use the present method to obtain k_1 and k_2. However, the presented exact warping functions for circular and rectangular isotropic cross sections can be used to check the accuracy of a two-dimensional sectional analysis code for analyzing general anisotropic beams.

6. Closure

A method of deriving exact shear warping functions of isotropic beams from elasticity solutions is shown in this paper. Moreover, a new derivation of shear correction factors is shown, in which the shear stress resultants and shear strain energy are conserved. The new shear correction factor is energy-consistent and its physical meaning is shown. The present shear correction factor is useful in finite-element formulations to simplify the formulation but still accurately account for transverse shear effects. The use of the energy-averaged shear representative in the derivation of Timoshenko's beam theory for planar and three-dimensional vibrations is also shown in detail.

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References

Bert, C.W., 1983. Simplified analysis of static shear correction factors for beams of nonhomogeneous cross section. Journal of Composite Materials 7, 525–529.

Chow, T.S., 1971. On the propagation of flexural waves in an orthotropic laminated plate and its response to an impulsive load. Journal of Composite Materials 5, 306–319.

Cowper, G.R., 1966. The shear coefficient in Timoshenko's beam theory. Journal of Applied Mechanics 33, 335–340.

Dong, S.B., Tso, F.K.W., 1972. On a laminated orthotropic shell theory including transverse shear deformation. Journal of Applied Mechanics 39, 1091–1097.

Giavotto, V., Borri, M., Mantegazza, P., Ghiringhelli, G., Carmaschi, V., Maffioli, G.C., Mussi, F., 1983. Anisotropic beam theory and applications. Computers and Structures 16, 403–413.

Iesan, D., 1987. St Venant's Problem. Lecture Notes in Mathematics, No. 1279, ed. A. Dold and B. Eckmann. Springer-Verlag, Berlin, Germany.

Love, A.E.H., 1944. A Treatise on the Mathematical Theory of Elasticity, 4th edn. Dover Publications, New York.

Muskhelishvili, N.I., 1963. Some Basic Problems of the Mathematical Theory of Elasticity, 4th edn, translated by J.R.M. Radok.P. Noordhoff Ltd, Groningen, The Netherlands.

Noor, A.K., Burton, W.S., 1989. Assessment of shear deformation theories for multi-layered composite plates. Applied Mechanics Review 42, 1–13.

Pai, P.F., 1995. A new look at shear correction factors and warping functions of anisotropic laminates. International Journal of Solids and Structures 32, 2295–2313.

Pai, P.F., Nayfeh, A.J., 1994. A fully nonlinear theory of curved and twisted composite rotor blades accounting for warpings and three-dimensional stress effects. International Journal of Solids and Structures 31, 1309–1340.

Pai, P.F., Nayfeh, A.H., Oh, K., Mook, D.T., 1993. A refined nonlinear model of composite plates with integrated piezoelectric actuators and sensors. International Journal of Solids and Structures 30, 1603–1630.

Reddy, J.N., Liu, C.F., 1985. A higher-order shear deformation theory of laminated elastic shells. International Journal of Engineering Science 23, 319–330.

Shames, I.H., Dym, C.L., 1985. Energy and Finite Element Methods in Structural Mechanics, p. 279. McGraw-Hill, New York.

Timoshenko, S.P., Goodier, J.N., 1970. Theory of Elasticity, 3rd edn. McGraw-Hill, New York.

Whitney, J.M., 1973. Shear correction factors for orthotropic laminates under static load. Journal of Applied Mechanics 40, 302–304.

Whitney, J.M., 1987. Structural Analysis of Laminated Anisotropic Plates. Technomic Publishing Company, Inc., Pennsylvania.

Yang, P.C., Norris, C.H., Stavsky, Y., 1966. Elastic wave propagation in heterogeneous plates. International Journal of Solids and Structures 2, 665–684.